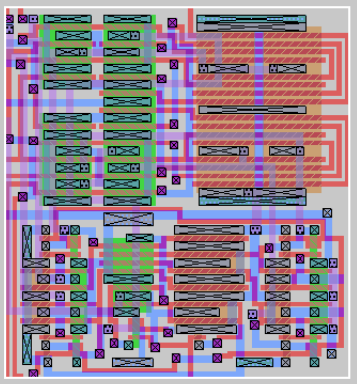
A Study of Bar and Arc $k-$Visibility Graphs

Mehtaab Sawhney
Under the Guidance of Jonathan Weed
5^{th} Annual MIT Primes Conference

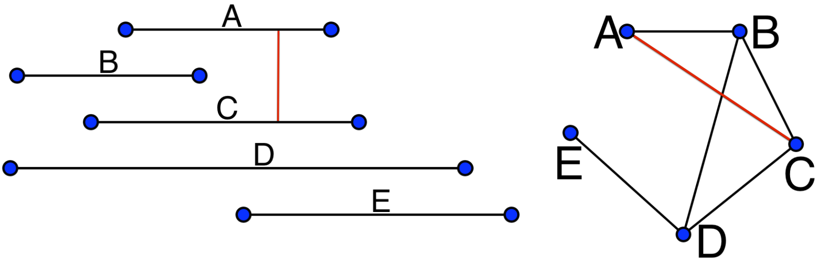
May 15, 2015

Motivation



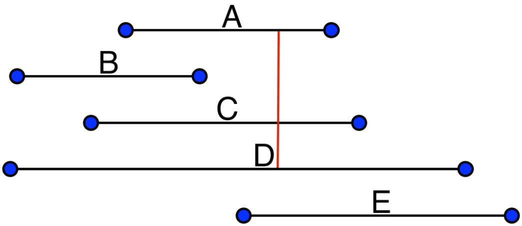
Bar Visibility Graph

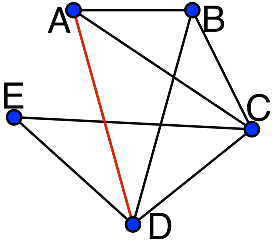
- An edge is drawn in a Bar Visibility Graph between two vertices if the corresponding bars have a direct line of sight.



Bar $k-$Visibility Graphs

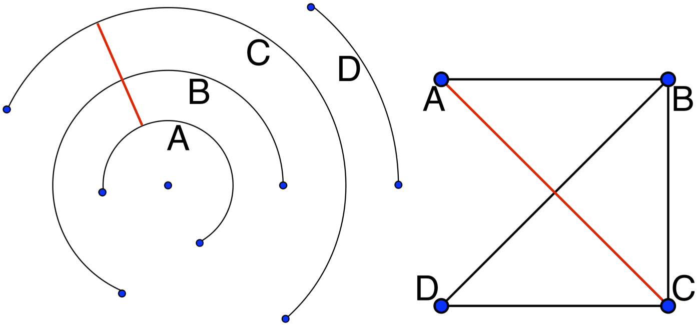
- [Dean, Evans, Gethner, Liaison, Safari, Trotter, 2007] introduced a generalization in which two vertices were connected if the corresponding bars have a line of sight that passes through $\leq k$ bars.





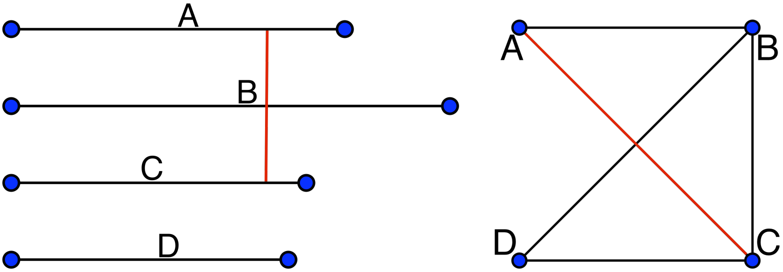
Arc $k-$Visibility Graphs

- An extension by [Hutchison, 2002] replaced bars with concentric circular arcs. [Babbit, Geneson, Khovanova, 2012] extended this idea to Arc $k-$Visibility Graphs.



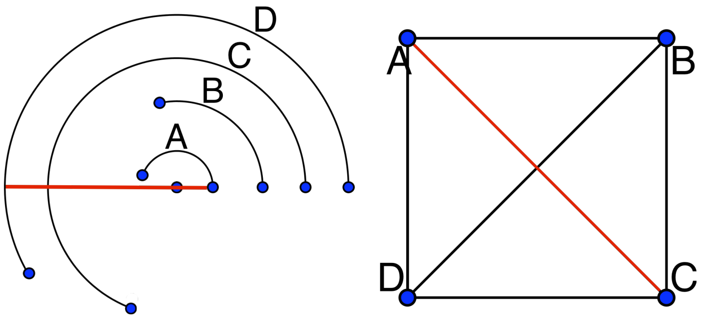
Semi-Bar $k-$Visibility Graphs

- In a Semi-Bar $k-$Visibility Graph all the bars share the same left endpoint.



Semi-Arc $k-$Visibility Graphs

- In a Semi-Arc $k-$Visibility Graph all the arcs share the same left endpoint.



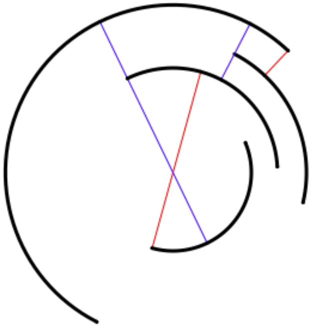
Summary of Edge Bounds

	A_k	SA_k
Babbit et al.	$\leq (k+1)(3n-k-2)$	$\leq (k+1)(2n-\frac{k+2}{2})$
Our Work	$\leq (k+1)(3n-k-3)$	

Table: Summary of Edge Bounds

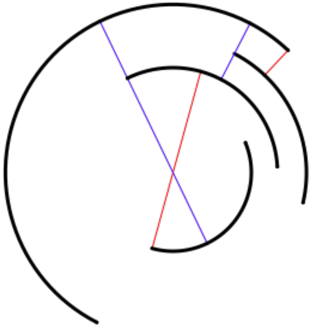
Positive and Negative Edges

- For a given set of arcs and the set of radial lines of sight take the infinum of each line of sight between the bars.
- If a radial line contains an endpoint of one of the two bars it a positive edge.
- If it contains the endpoint of another bar it is a negative edge.



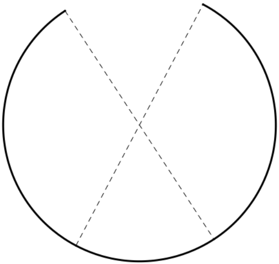
Edge Bound for Arc k-Visibility Graphs

- [Babbit, Geneson, Khovanova, 2012] gave a bound of $(k+1)(3n-k-2)$ edges for $n > 4k+4$ and $\binom{n}{2}$ for $n \leq 4k+4$.
- For the bound to be tight the outermost $k+1$ arcs radially have $k+1, k+2, \ldots, 2k+1$ positive edges and $0, 1, \ldots, k$ negative edges respectively.



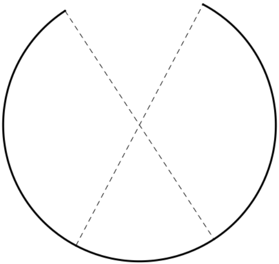
Improved Edge Bound for Arc Visibility Graphs

- Consider the outermost bar and the "cones" created by gap in the outermost arc.



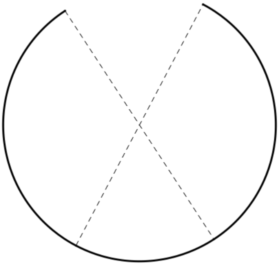
Improved Edge Bound for Arc Visibility Graphs

- Consider the outermost bar and the “cones” created by gap in the outermost arc.
- If no edges lie within in the cone there is a missing edge for the outermost arc.



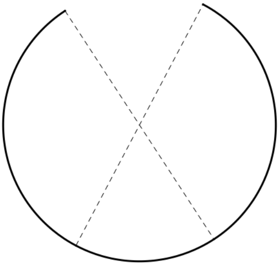
Improved Edge Bound for Arc Visibility Graphs

- Consider the outermost bar and the "cones" created by gap in the outermost arc.
- If no edges lie within in the cone there is a missing edge for the outermost arc.
- If there is an edge that is only partially in the arc it is missing an edge.



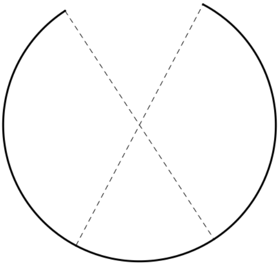
Improved Edge Bound for Arc Visibility Graphs

- Consider the outermost bar and the "cones" created by gap in the outermost arc.
- If no edges lie within in the cone there is a missing edge for the outermost arc.
- If there is an edge that is only partially in the arc it is missing an edge.
- If an edge covers the entire arc an edge is double counted as both a positive and negative edge for the inner arc.

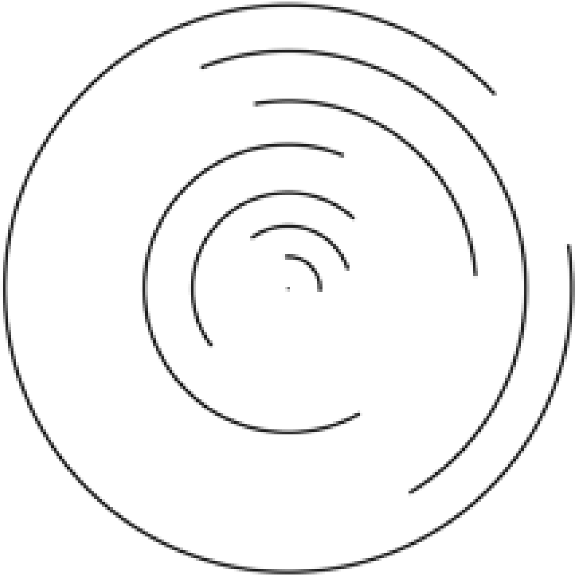


Improved Edge Bound for Arc Visibility Graphs

- Consider the outermost bar and the "cones" created by gap in the outermost arc.
- If no edges lie within in the cone there is a missing edge for the outermost arc.
- If there is an edge that is only partially in the arc it is missing an edge.
- If an edge covers the entire arc an edge is double counted as both a positive and negative edge for the inner arc.
- Using a similar method the edge bound can be improved by $(k+1)$ edges in the general case.



Improved Edge Bound for Arc Visibility Graphs (cont.)



Thickness

- Thickness, $\theta(G)$ is the minimal number of graphs into which G is partitioned such that each is planar.
- Thickness does not hold under minors.

Summary of Thickness Bounds

	$\theta(B_k)$	$\theta(SB_k)$	$\theta(A_k)$	$\theta(SA_k)$
Dean et al.	$\leq 3k(6k+1)$			
Chang et al.	$\leq 3k+3$	$\leq 2k$		
Babbit et al.		$\leq 2k$		
Our Work			$\leq 3k+3$	$\leq 2k+2$

Table: Summary of Thickness Bounds

Arboricity and Nash-Williams Theorem

- Arboricity is the minimal number of trees for which a graph can be partitioned into.
- For any graph G it follows that $\theta(G) \leq arb(G)$.
- By Nash-Williams Theorem $arb(G) = \max_{H \subseteq G} \left\lceil \frac{E_H}{N_H - 1} \right\rceil$.

Thickness Bound Theorem on Arc $k-$Visibility Graphs

- Notice that every subgraph of an Arc k-Visibility Graph with ℓ vertices is a subgraph of Arc k-Visibility Graph with ℓ vertices.
- This lemma comes from [Chang, Lu, Sung, 2010].

Thickness Bound Theorem on Arc $k-$Visibility Graphs (cont.)

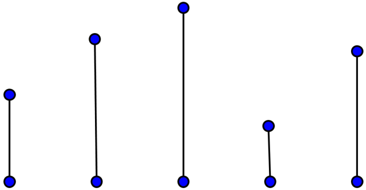
- Thus by Nash-Williams it follows that $arb(G) = \max_{H \subseteq G} \left\lceil \frac{E_H}{N_H - 1} \right\rceil$.
- If $N_H \leq 4k + 4$ then $\left\lceil \frac{E_H}{N_H - 1} \right\rceil \leq 2k + 2$.
- If $N_H > 4k + 4$ then $\left\lceil \frac{E_H}{N_H - 1} \right\rceil \leq 3k + 3$.
- Thus it follows that $arb(G) \leq 3k + 3$.

Theorem on Thickness of Semi-Arc $k-$Visibility Graph

- Notice that each line of sight can be adjusted so that each line of sight contains one of the endpoints. Direct edges from the bar with the endpoint to the one without.
- It follows that $d_+(v) \leq 2k + 2$ for any vertex in the graph.
- Therefore it follows that $\theta(G) \leq 2k + 2$.

Theorem on Expected Number of Edges in a Random Semi-Bar $k-$Visibility Graph

- If there are $\leq k+2$ edges then there is always a complete graph so $E_n^k = \binom{n}{2}$ for $n \leq k+2$.
- If E_n^k is known then for $n+1$ bars consider the shortest bar. Removing the shortest bar the remaining graph still has E_n^k edges. For the shortest bar count the expected number of edges the shortest bar can see to the right and then to the left.
- Then $E_n^k = \frac{1}{2}(k+1)(4n-3k-6-\sum_{l=k+3}^{n}\frac{1}{l})$ for $n \geq k+3$.



Goals

- What is the tight edge bound for Arc $k-$Visibility Graphs?
- To develop better thickness bounds for Bar $k-$Visibility Graphs and Arc $k-$Visibility Graphs.
- To calculate the expected number of edges in a Bar $k-$Visibility Graphs.

Acknowledgements

- I would like to sincerely thank Jonathan Weed for his guidance.
- I would like to express my gratitude to the MIT PRIMES Program for the opportunity.
- I would like to thank my parents, as always, for their support.

References I

Alice Dean, William Evans, Ellen Gethner, Joshua D. Laison, Mohammad Ali Safari, and William T. Trotter. Bar k-visibility graphs: Bounds on the number of edges, chromatic number, and thickness. Journal of Graph Algorithms and Applications, 11:45-49, 2007.

Stefan Felsner and Mareike Massow. Parameters of bar k-visibility graphs. Journal of Graph Algorithms and Applications, 12:5-27, 2008.

Stephen Hartke, Jennifer Vandenbussche, and Paul Wenger. Further results on bar k-visibility graphs. SIAM Journal on Discrete Mathematics, 21:523-531, 2007.

Joan P. Hutchinson. Arc- and circle-visibility graphs. Australasian Journal of Combinatorics, 25:241-262, 2002.

References II

Petra Mutzel, Thomas Odenthal, and Mark Scharbrodt. The thickness of graphs: a survey. Graphs and Combinatorics, 14:59-73, 1998.

Y. Sung, Hsueh-I Lu, and Yen-Peng Sung. Asymptotically optimal thickness bounds of generalized bar visibility graphs. NTU Institutional Repository, 2009. http://ntur.lib.ntu.edu.tw/handle/246246/185442.

Perturbations

- When edge and thickness edge bounds are being considered bars and arcs can be perturbed such that no two bars or arcs need to have the same endpoints.